Theory of Observations

Trajectories of photons = null geodesics
Observer at $(r = 0, \theta, \phi, t = t_0)$
emitting light source at $(r_\mathrm{S}, \theta = 0, \phi = 0, t_\mathrm{S})$
$r(t)$ be the trajectory of the photons emitted. As this trajectory is a null geodesic, we have:

$$c^2 dt^2 - R^2(t)\frac{dr^2}{1 - kr^2} = 0$$

i.e.

$$\frac{cdt}{R(t)} = \frac{dr}{\sqrt{1 - kr^2}}$$

General Mattig relation

$$\int_{t_S}^{t_0} \frac{c\,dt}{R(t)} = \int_0^{r_S} \frac{dr}{(1-kr^2)^{1/2}} = S_k^{-1}(r_S)$$

with:

$$S_k(r_S) = \begin{cases} \sin(r_S) & \text{if } k = +1 \\ r_S & \text{if } k = 0 \\ \sinh(r_S) & \text{if } k = -1 \end{cases}$$

When the distance is small in front of R_0 we just have:

$$S_k^{-1}(r) \sim r \text{ and } l.h.s. \sim \frac{c\delta t}{R(t_0)}$$

The redshift

A source emitting at the frequency ν_S is observed at frequency ν_0

We consider the two trajectories of the light ray emitted at the time $t_{\rm S}$ and $t_{\rm S} + \frac{1}{\nu_S}$ arriving at t_0 and $t_0 + \frac{1}{\nu_0}$

The comoving coordinate $r_{\rm S}$ of the source remains constant so:

$$S_k^{-1}(r_{\rm S}) = \int_{t_{\rm S}}^{t_0} \frac{cdt}{R(t)} = \int_{t_{\rm S}+1/\nu_S}^{t_0+1/\nu_0} \frac{cdt}{R(t)}$$

The redshift (2)

so:

$$\frac{c}{R(t_0)}\frac{1}{\nu_0} - \frac{c}{R(t_S)}\frac{1}{\nu_S} = 0$$

leading to the *redshift* z:

$$1 + z = \frac{\nu_s}{\nu_0} = \frac{\lambda_0}{\lambda_E} = \frac{R_0}{R_S}$$

Interpretation?

Doppler?
Gravitational?
(It is not the same!)

The redshift (3)

If the “distance” changes with time:

$$v = \frac{\Delta l}{\Delta t}$$

and if :

$$\frac{\Delta\lambda}{\lambda} = \frac{v}{c} \text{ (first order)}$$

this could be qualified as a purely Doppler shift.

The proper distance

Distance obtained as a sum of rulers:

$$dl^2 = -ds^2 = R(t)^2 \frac{dr^2}{1 - kr^2}$$

so that the proper distance is :

$$D = \int_0^S dl = R(t) S_k^{-1}(r_\mathrm{S})$$

This distance varries with time:

$$\dot{D} = \dot{R} S_k^{-1}(r_S)$$

The proper distance

so that the source is receding at speed:

$$v = \frac{\dot{R}}{R} D = HD$$

This is the Hubble law.

The redshift from : $\frac{\nu_0}{\nu_s} = \frac{R(t_S)}{R(t_0)} \sim \frac{R(t_0)+\dot{R}(t_S-t_0)}{R(t_0)}$

so that:

$$\frac{\nu_S - \nu_0}{\nu_s} = \frac{\delta\nu}{\nu} = \frac{\dot{R}}{R}\delta t = H\frac{D}{c} = \frac{v}{c}$$

so it is a **Doppler shift.**

Distances...

when $r << 1$ space can be regarded as beeing flat.

i.e. $R(t_s) \sim R(t_0)$ or

$$z << 1$$

when $z \geq 1$ this is not true anymore

A “distance measurement” needs a precise experimental procedure.

Different procedures lead to different answers.

Distances...

- Angular distance : $\theta = \frac{d}{D}$
- Luminosity distance : $l = \frac{L}{4\pi D^2}$
- Paralax distance : $\pi = \frac{R_T}{D}$
- ...

Angular distance

Take a ruler : size d seen from epoch t_{S}
Observer: $(r = 0, 0, 0, t = t_0)$
ruler: $(r_{\mathrm{S}}, 0, 0, t_{\mathrm{S}})$ and $(r_{\mathrm{S}}, \theta, 0, t_{\mathrm{S}})$
Proper length:

$$d^2 = -ds^2 = R^2(t_S) r^2 \theta^2$$

by definition: $\theta = \frac{d}{D_{\mathrm{ang}}} = \frac{d}{R(t_S) r}$

so:

$$D_{\mathrm{ang}} = R(t_S) r$$

Surface brightness

Energy going through a surface dA, during dt, in the frequency range $\nu, \nu + d\nu$:

$$du = i(\nu)\, d\nu\, dA\, dt\, d\Omega$$

i : specific intensity.
In terms of the distribution function of photons:

$$du = f(p)\, p^2\, dp\, d\Omega\, dA\, cdt\, pc$$

($p = h\nu/c$)
so:

$$i(\nu) \propto f(p)\, p^3$$

Surface brightness

Liouville theorem: $f(p)$ is conserved during propagation, so:

$$\frac{i(\nu)}{p^3} \propto \frac{i(\nu)}{\nu^3} = \mathrm{cste\ i.e.\ i}(\nu_0) = \frac{\mathrm{i}(\nu_\mathrm{S})}{(1+\mathrm{z})^3}$$

$$\frac{\nu_S}{\nu_0} = 1 + z = \frac{d\nu_S}{d\nu_0}$$

Integrated surface brightness:

$$\int_0^{+\infty} i(\nu_0)\, d\nu_0 = \frac{1}{(1+z)^4} \int_0^{+\infty} i(\nu_S)\, d\nu_S$$

test of expansion (Tolman, 1931; Sandage and Perulmuter, 1991)

Luminosity distance

Telescope with diameter $2d$ observes a point source of luminosity L
2θ is the angle of the telescope *seen from the source*

$$d = R(t_0)\; r\; \theta$$

l: the apparent luminosity of the source

$$l = L\,\frac{\pi\theta^2}{4\pi}\,\frac{1}{1+z}\,\frac{1}{1+z}\,\frac{1}{\pi\, d^2}$$

$$l = \frac{L}{4\pi(R(t_0)\; r)^2}\,\frac{1}{(1+z)^2} = \frac{L}{4\pi\; D_{\mathrm{lum}}^2}$$

Luminosity distance

$$
\begin{aligned}
D_{\mathrm{lum}} &= R(t_0)\, r\, (1+z) \\
&= R(t_S)\, r\, (1+z)^2 \\
&= D_{\mathrm{ang}}\, (1+z)^2
\end{aligned}
$$

This last relation is always valid.

Distance along the line of sight: $dl = c dt$
$1 + z = R_0/R(t)$ one gets $dz = -H(z)(1+z)dt$ so:

$$
dl = -\frac{c}{H(z)}\,\frac{dz}{1+z}
$$

Volume element

$$
\begin{aligned}
dV &= d\Omega\, D_{ang}^2\, dl \\
&= -d\Omega (R(t_E)\, r)^2 \frac{c}{H(z)}\, \frac{dz}{1+z} \\
&= -d\Omega\, (R(t_E)\, r)^2 R(t_E) \frac{dr}{\sqrt{1-kr^2}}
\end{aligned}
$$

$\rightarrow$ useful for number counts.