Dynamics

based on Einstein's G.R.

- $R_{ij} - 1/2 g_{ij} R = 8\pi G T_{ij}$
- rest frame :

$$T_{ij} = \begin{bmatrix} \rho & & & \\ & P & & \\ & & P & \\ & & & P \end{bmatrix}$$

- Source of gravity : $\rho + 3P/c^2$
- Birkoff's theorem : analog of Gauss theorem

For spherical distribution only $\rho(r < R)$ matters for the solution within $r < R$.

Solutions

Inside a sphere of Radius a

$$\ddot{a} = g$$

Source is $\rho + 3P/c^2$:

$$\ddot{a} = -\frac{GM}{a^2} = -\frac{4\pi G}{3}(\rho + 3P/c^2)a \qquad (1)$$

Energy conservation

E_t total energy of the sphere :

$$\begin{aligned} d(E_t) &= d(\rho V c^2) = -PdV \\ &= c^2(V d\rho + \rho dV) = -PdV \end{aligned}$$

Solutions

leading to :

$$\dot{\rho} = -(\rho + P/c^2)\frac{\dot{V}}{V} = -3(\rho + P/c^2)\frac{\dot{a}}{a} \qquad (2)$$

(1) and (2) allow to eliminate P:

$$\ddot{a} = -\frac{4\pi G}{3}(\rho + 3P/c^2)a$$

$$\ddot{a} = -\frac{4\pi G}{3}(3\rho + 3P/c^2)a + 2\frac{4\pi G}{3}\rho a$$

$$\ddot{a} = +\frac{4\pi G}{3}\frac{a\dot{\rho}}{\dot{a}}a + 2\frac{4\pi G}{3}\rho a$$

$$\dot{a}\ddot{a} = +\frac{8\pi G}{3}\rho a\dot{a} + \frac{4\pi G}{3}a^2\dot{\rho}$$

$$(\dot{a}^2)' = \left(\frac{8\pi G a^2\rho}{3}\right)'$$

that is :

$$\dot{a}^2 = \frac{8\pi G a^2\rho}{3} + cste$$

For $R(t)$:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{R^2}$$

Equation of state

Solution -> needs an equation of state $F(\rho, P) = 0$

Notation : $P = w\rho$

The density ρ reads:

$$\rho = \sum_i \int \frac{E_i}{c^2} f(p_i) dp_i$$

the pressure P:

$$P = \sum_i \int \frac{1}{3} \frac{p_i^2}{E_i} f(p_i) dp_i$$

Equation of state

Two important regimes:

$\longrightarrow$ matter dominated: $p << mc$ i.e. $P = 0$

$\rho = \int m$ and $g \propto \rho$

$\dot{\rho} = -3\rho\dot{a}/a$ ($a \propto R$) so :

$$\rho a^3 = \mathrm{cste}$$

$\longrightarrow$ pressure (radiation) dominated:

$p >> mc$ so $\rho = \int p/c...$ and $P = \int 1/3\, p\, c...$

$$P = \frac{1}{3}\rho c^2$$

$\dot{\rho} = -4\rho\dot{a}/a$ so :

$$\rho a^4 = \mathrm{cste}$$

Vacuum I

Naively : $\rho_v = 0$ and $P_v = 0$
But take a box with vacuum in it:

$$d(E_t) = d(\rho_v V c^2) = \rho_v c^2 dV = -P_v dV$$

so we get the equation of state of vacuum:

$$P_v = -\rho_v c^2$$

Introducing the cosmological constant:

$$\Lambda = 8\pi G \rho_v$$

Vacuum II

From quantum field point of view
Harmonic oscillator:

$$E_n = (n + \frac{1}{2})h\nu$$

zero point energy: $\frac{1}{2}h\nu$
zero point energies of fields contributes to ρ_V:

$$\rho_V = \int_0^{k_c} \frac{4\pi k^2 dk}{8\pi^3} \frac{1}{2} \sqrt{k^2 + m^2} \sim \frac{{k_c}^4}{8\pi^2}$$

for $k_c \gg m$.

Quintessence

$$P/c^2 = w\rho$$

For a scalar field, Φ, the density is:

$$\rho_\Phi = \frac{1}{2}\dot{\Phi}^2 + V(\Phi)$$

and the pressure P:

$$P_\Phi/c^2 = \frac{1}{2}\dot{\Phi}^2 - V(\Phi)$$

Allowing $-1 \leq w \leq 0$. even $w \leq -1$ is possible...

Summary

Einstein-Friedmann-Lemaître (EFL) equations:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{R^2} + \frac{\Lambda}{3}$$

and

$$\dot{\rho} = -3\left(\frac{P}{c^2} + \rho\right)\frac{\dot{R}}{R}$$

$$2\frac{\ddot{R}}{R} = -\frac{8\pi G}{3}(\rho + 3P/c^2) + \frac{2\Lambda}{3}$$

Notations

$H = \frac{\dot{R}}{R}$, the Hubble parameter,
$\Omega_M = \Omega = \frac{8\pi G\rho}{3H^2}$ the density parameter,
$q = -\frac{\ddot{R}R}{\dot{R}^2}$, the deceleration parameter,
$\Omega_{\text{vac}} = \Omega_\lambda = \lambda = \frac{\Lambda}{3H^2}$, the (reduced) cosmological constant,
$\Omega_c = \alpha = \frac{kc^2}{H^2R^2}$, the curvature parameter.
Quantities are labeled by 0 when they are referred to their present value: Ω_0, q_0, ...
E.F.L. :

$$\Omega_c = \Omega_M + \Omega_\lambda - 1$$

Solutions (matter dominated)

$$\ddot{a} = g = -\frac{GM}{a^2} \text{ and } + \ \rho\, a^3 = \text{cste}$$

from this we have derived:

$$\dot{a}^2 - \frac{8\pi\, G\, \rho\, a^2}{3} = \dot{a}^2 - \frac{2GM}{a} = -k\, c^2$$

This is exactly the equation of a test particle in the field of one mass in Newtonian theory!

$$E_c + E_p = cste$$

Solutions:

- k = -1 unbound hyperbolic solution
- k = 0 parabolic solution
- k = +1 bound elliptic solution

$k = 0$ ($P = 0$)

$$\dot{R}^2 = \frac{8\pi\ G\ \rho\ R^2}{3} \text{ and } \rho R^3 = \rho_0 R_0^3$$

First Eq. implies:

$$\Omega = \frac{8\pi\ G\ \rho}{3\ H^2} = 1 = \Omega_0$$

(present-day) critical density :

$$\rho_c = \frac{3\ H_0^2}{8\pi\ G}$$

Second Eq. implies:

$$\dot{R}^2 = \frac{8\pi\ G\ \rho_0\ R_0^3}{3\ R} = H_0^2 \frac{R_0^3}{R}$$

$k = 0$ ($P = 0$) Solution:

$$R(t) = R_0 \left(\frac{3}{2} H_0 \, t \right)^{2/3} = R_0 \, (t/t_0)^{2/3}$$

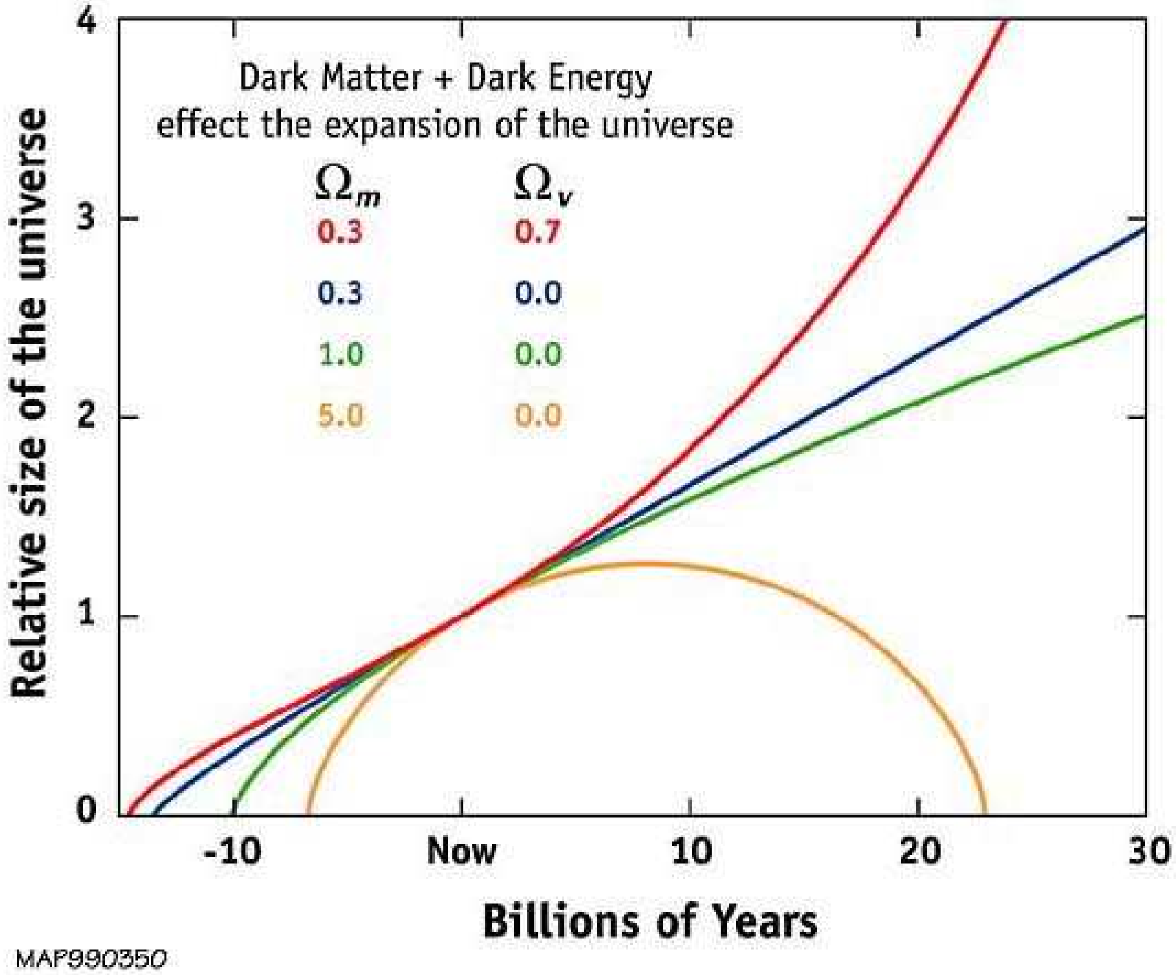
with :

$$t_0 = \frac{2}{3} \, H_0^{-1} = \frac{1}{\sqrt{6\pi \, G \, \rho_c}}$$

This solution goes through 0 in the past...

The solution has an “Initial” singularity.

Solutions:



Dark Matter + Dark Energy
effect the expansion of the universe
Ω_m Ω_v
0.3 0.7
0.3 0.0
1.0 0.0
5.0 0.0
Relative size of the universe
4
3
2
1
0
-10
Now
10
20
30
Billions of Years
MAP990350

General behavior:

$$2\frac{\ddot{R}}{R} = -\frac{8\,\pi\,G}{3}(\rho + 3P/c^2)$$

and :

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\,\pi\,G\,\rho}{3} - \frac{k\,c^2}{R^2}$$

so if: $(\rho + 3P/c^2) > 0$ R will go through 0 (in the past) in a finite time t_0.

There is a theorem more general than this.

When $R \to 0$ than $\left(\frac{\dot{R}}{R}\right)^2 \sim \frac{8\,\pi\,G\,\rho}{3}$ i.e. $\Omega \sim 1$

Behavior of Ω :

previous second Eq. implies $\Omega_c = \Omega - 1$ so :

$$H^2 = H_0^2[\Omega_0(1+z)^3 + (1-\Omega_0)(1+z)^2]$$

so:

$$H^2 = H_0^2(1+z)^2(1+\Omega_0 z)$$

and:

$$\Omega(z) = \frac{8\pi\, G\, \rho}{3H^2} = \frac{8\pi\, G\, \rho_0}{3H_0^2}\frac{(1+z)^3}{(1+z)^2(1+\Omega_0 z)}$$

so:

$$\Omega(z) = \Omega_0\frac{(1+z)}{(1+\Omega_0 z)}$$

Mattig relation :

Along a light ray:

$$\frac{dr^2}{1-kr^2} = \frac{c^2 dt^2}{R^2(t)} = \frac{c^2 dR^2}{R^2(t)\dot{R}^2(t)}$$

From this, setting $v = \frac{\alpha_0}{\Omega_0 R_0}\, R$ in the right hand side, one can derive (...):

$$R_0\, r = \frac{c}{H_0}\,\frac{2}{\Omega_0^2}\,\frac{\Omega_0(1+z)+2-2\Omega_0-(2-\Omega_0)\sqrt{1+\Omega_0\, z}}{1+z}$$

when $z << 1$ $R_0\, r \sim \frac{c}{H_0} z$

when $z >> 1$ $R_0\, r \sim \frac{c}{H_0}\frac{2}{\Omega_0}$

$k = -1$ ($P = 0$) Solution:

$$
\begin{aligned}
\dot{R}^2 &= \frac{8\pi\, G\, \rho R^2}{3} - kc^2 \\
&= H_0^2 \Omega_0\, R_0^2\, (1+z) + (1-\Omega_0)\, H_0^2\, R_0^2
\end{aligned}
$$

so when $1 + z >> \frac{1-\Omega_0}{\Omega_0}$ one has : $R \propto t^{2/3}$

while when $1 + z << \frac{1-\Omega_0}{\Omega_0}$ $\dot{R} \sim cste$ one has $R \propto t$ $R(t)$ can be developped:

$$
\begin{aligned}
H_0\, t &= \frac{\Omega_0}{2(1-\Omega_0)^{3/2}} (\sinh(\psi) - \psi) \\
\frac{1}{1+z} &= \frac{R(t)}{R_0} = \frac{\Omega_0}{2(1-\Omega_0)} (\cosh(\psi) - 1)
\end{aligned}
$$

Allows analytical expression of $H_0\, t(z)$

$k = +1$ ($P = 0$) Solution:

The expression:

$$\dot{R}^2 = H_0^2 \Omega_0 \, R_0^2 \, (1+z) + (1-\Omega_0) \, H_0^2 \, R_0^2$$

allows to find R_m so that $\dot{R} = 0$

$$R_m = R_0 \, \frac{\Omega_0}{\Omega_0 - 1}$$

$R(t)$ can be developped as well:

$$H_0 \, t = \frac{\Omega_0}{2(\Omega_0 - 1)^{3/2}} (\phi - \sin(\phi))$$

$$\frac{1}{1+z} = \frac{R(t)}{R_0} = \frac{\Omega_0}{2(\Omega_0 - 1)} (1 - \cos(\phi))$$

$k = +1$ ($P = 0$) Solution:

At the maximum:

$$
\begin{aligned}
R_m &= c \frac{2\, t_m}{\pi} \\
\rho_m &= \frac{3\pi}{32\, G\, t_m^2} \\
t_m &= \frac{1}{H_0} \frac{\Omega_0}{(\Omega_0 - 1)^{3/2}}\, \pi
\end{aligned}
$$

(useful for structure formation)

Cases $\Lambda \neq 0$

$$2\ddot{R} = -\frac{8\,\pi G}{3}(\rho + \frac{3P}{c^2})R + \frac{2\Lambda}{3}R$$

If $\Lambda < 0$ it is an attractive force

If $\Lambda > 0$ it is a repulsive force, in which case $R(t)$ might not go through $R = 0$.

Case $P = 0$

$$2\ddot{R} = H_0^2 R_0 [\frac{2\;\lambda_0}{(1+z)} - \Omega_0\;(1+z)^2]$$

$$\dot{R}^2 = H_0^2 R_0^2 [\frac{\lambda_0}{(1+z)^2} + (1 - \Omega_0 - \lambda_0) + \Omega_0 (1+z)]$$

setting $u = 1 + z$ one gets:

$$\dot{R}^2 \propto \frac{\lambda_0}{u^2} + (1 - \Omega_0 - \lambda_0) + \Omega_0\;u = f(u)$$

Case $\Lambda \neq 0, P = 0$

The "useful" relations $R_0 r$, $t(z)$, ... are not analytical.

$$\begin{aligned} \dot{R}^2 &= \frac{8\,\pi G\,\rho\,R^2}{3} - kc^2 + \frac{\Lambda R^2}{3} \\ &= H_0^2 R_0^2 [\frac{\Omega_\Lambda}{(1+z)^2} - \Omega_c + \Omega_0 (1+z)] \end{aligned}$$

Mattig relation

$$S_k^{-1}(r) = \int_{t(z)}^{t_0} \frac{c\,dt}{R(t)} = |\Omega_c|^{1/2} \int_1^{1+z} \frac{d\,u}{(\Omega_0 u^3 - \Omega_c u^2 + \Omega_\Lambda)^{1/2}}$$

Age:

$$t_0 - t(z) = \int_1^{1+z} \frac{1}{H_0} \frac{d\,u}{u(\Omega_0 u^3 - \Omega_c u^2 + \Omega_\Lambda)^{1/2}}$$

Case $\Lambda \neq 0$ Applications

- Mattig relation : $R_0\, r(z)$
- Angular distance : $\theta = \frac{d}{D_{ang}(z)}$
  $\rightarrow$ minimum at some z then increases!
- Look back time: $H_0(t_0 - t(z))$

$\rightarrow$ at $z \sim 1$ the universe is significantly younger:

$$\Omega \sim 0. \quad \Omega_\Lambda = 0. \qquad z = 1 \leftrightarrow t_1 \sim 0.5 \quad t_0$$

$$\Omega = 1. \quad \Omega_\Lambda = 0. \qquad z = 1 \leftrightarrow t_1 \sim 0.35\, t_0$$

$$\Omega = 0.3\; \Omega_\Lambda = 0.7 \qquad z = 1 \leftrightarrow t_1 \sim 0.35\, t_0$$

Models with $(\Omega, \Omega_\Lambda > 0)$ are older than with $(\Omega, \Omega_\Lambda = 0)$, the difference being important only when $\Omega_\Lambda \sim \lambda_c$.

Radiation dominated

$$P = \frac{1}{3}\rho_\gamma \, c^2 \quad \text{and} \quad \rho_\gamma \, R^4 = \text{cste}$$

E.F.L. Equations:

$$\begin{aligned} \frac{\dot{R}}{R}^2 &= \frac{8\,\pi G}{3}(\rho_\gamma + \rho_m) - \frac{kc^2}{R^2} + \frac{\Lambda}{3} \\ &\propto \quad \frac{1}{R^4} \quad \frac{1}{R^3} \quad \frac{1}{R^2} \quad \text{cste} \end{aligned}$$

$\longrightarrow$ The radiation term is dominant at high redshift: $\dot{R} = \frac{\text{cste}}{R}$

Solution:

$$R = R_1 \left(\frac{t}{\tau}\right)^{1/2} \quad \text{with} \quad \tau = \frac{3}{32\,\pi\,G\,\rho_1}$$

Some historical remarks

Einstein: 1916: GR + first consistent cosmological model.
Einstein cosmological principle: The universe is homogeneous on large scale.

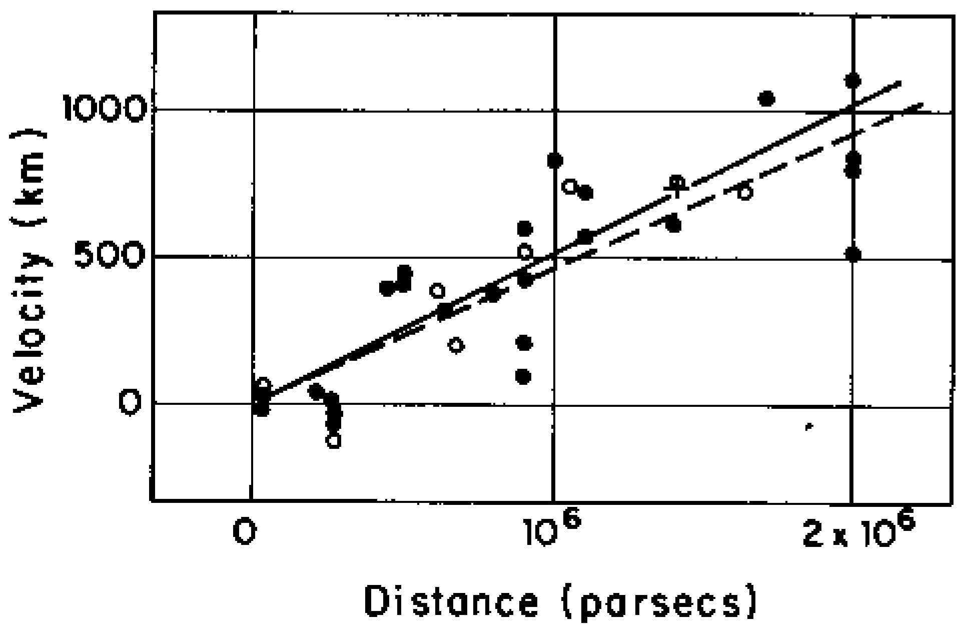
De Sitter: 1919 GR $+\Lambda$ with $\rho = 0$.
Static but particles move. Redshft $\propto D$.

Friedman: 1922-1924: G.R. general solutions with positive and negative curvature. Polemic with Einstein.

Some historical remarks

Lemaître 1925: De Sitter world = expanding world.
1927: expanding solution with $\rho \neq 0$.

Hubble 1929: The linear relation between D and v



1933

Zwicky Missing mass in Coma.

Lemaître Beginning ? Singularity ? How did structures originate ?

Gamov 1942-1948: Origin of elements $\longrightarrow T$

Penzias, Wilson, Dicke's group 1964: Discovery and interpretation of the CMB.