

# Dynamics

based on Einstein's G.R.

- $R_{ij} - 1/2g_{ij}R = 8\pi GT_{ij}$
- rest frame :  $T_{ij} = \begin{bmatrix} \rho & & & \\ & P & & \\ & & P & \\ & & & P \end{bmatrix}$
- Source of gravity :  $\rho + 3P/c^2$
- Birkoff's theorem : analog of Gauss theorem

For spherical distribution only  $\rho(r < R)$  matters for the solution within  $r < R$ .

# Solutions

Inside a sphere of Radius  $a$

$$\ddot{a} = g$$

Source is  $\rho + 3P/c^2$ :

$$\ddot{a} = -\frac{GM}{a^2} = -\frac{4\pi G}{3}(\rho + 3P/c^2)a \quad (1)$$

Energy conservation

$E_t$  total energy of the sphere :

$$\begin{aligned} d(E_t) &= d(\rho V c^2) = -P dV \\ &= c^2(V d\rho + \rho dV) = -P dV \end{aligned}$$

# Solutions

leading to :

$$\dot{\rho} = -(\rho + P/c^2) \frac{\dot{V}}{V} = -3(\rho + P/c^2) \frac{\dot{a}}{a} \quad (2)$$

(1) and (2) allow to eliminate  $P$ :

$$\ddot{a} = -\frac{4\pi G}{3}(\rho + 3P/c^2)a$$

$$\ddot{a} = -\frac{4\pi G}{3}(3\rho + 3P/c^2)a + 2\frac{4\pi G}{3}\rho a$$

$$\ddot{a} = +\frac{4\pi G}{3} \frac{a\dot{\rho}}{\dot{a}} a + 2\frac{4\pi G}{3}\rho a$$

$$\ddot{a} = +\frac{8\pi G}{3}\rho a\dot{a} + \frac{4\pi G}{3}a^2\dot{\rho}$$

$$(\dot{a}^2)' = \left( \frac{8\pi G a^2 \rho}{3} \right)'$$

that is :

$$\dot{a}^2 = \frac{8\pi G a^2 \rho}{3} + cste$$

For  $R(t)$ :

$$\left( \frac{\dot{R}}{R} \right)^2 = \frac{8\pi G \rho}{3} - \frac{kc^2}{R^2}$$

# Equation of state

Solution  $\rightarrow$  needs an equation of state  $F(\rho, P) = 0$

Notation :  $P = w\rho$

The density  $\rho$  reads:

$$\rho = \sum_i \int \frac{E_i}{c^2} f(p_i) dp_i$$

the pressure  $P$ :

$$P = \sum_i \int \frac{1}{3} \frac{p_i^2}{E_i} f(p_i) dp_i$$

# Equation of state

Two important regimes:

→ matter dominated:  $p \ll mc$  i.e.  $P = 0$

$$\rho = \int m \text{ and } g \propto \rho$$

$$\dot{\rho} = -3\rho\dot{a}/a \text{ (} a \propto R \text{) so :}$$

$$\rho a^3 = \text{cste}$$

→ pressure (radiation) dominated:

$$p \gg mc \text{ so } \rho = \int p/c \dots \text{ and } P = \int 1/3 p c \dots$$

$$P = \frac{1}{3}\rho c^2$$

$$\dot{\rho} = -4\rho\dot{a}/a \text{ so :}$$

$$\rho a^4 = \text{cste}$$

# Vacuum I

Naively :  $\rho_v = 0$  and  $P_v = 0$

But take a box with vacuum in it:

$$d(E_t) = d(\rho_v V c^2) = \rho_v c^2 dV = -P_v dV$$

so we get the equation of state of vacuum:

$$P_v = -\rho_v c^2$$

Introducing the cosmological constant:

$$\Lambda = 8\pi G \rho_v$$

# Vacuum II

From quantum field point of view  
Harmonic oscillator:

$$E_n = \left(n + \frac{1}{2}\right)h\nu$$

zero point energy:  $\frac{1}{2}h\nu$

zero point energies of fields contributes to  $\rho_V$ :

$$\rho_V = \int_0^{k_c} \frac{4\pi k^2 dk}{8\pi^3} \frac{1}{2} \sqrt{k^2 + m^2} \sim \frac{k_c^4}{8\pi^2}$$

for  $k_c \gg m$ .



# Quintessence

$$P/c^2 = w\rho$$

For a scalar field,  $\Phi$ , the density is:

$$\rho_{\Phi} = \frac{1}{2}\dot{\Phi}^2 + V(\Phi)$$

and the pressure  $P$ :

$$P_{\Phi}/c^2 = \frac{1}{2}\dot{\Phi}^2 - V(\Phi)$$

Allowing  $-1 \leq w \leq 0$ . even  $w \leq -1$  is possible...

# Summary

Einstein-Friedmann-Lemaître (EFL) equations:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{R^2} + \frac{\Lambda}{3}$$

and

$$\dot{\rho} = -3\left(\frac{P}{c^2} + \rho\right)\frac{\dot{R}}{R}$$

$$2\frac{\ddot{R}}{R} = -\frac{8\pi G}{3}(\rho + 3P/c^2) + \frac{2\Lambda}{3}$$

# Notations

$H = \frac{\dot{R}}{R}$ , the Hubble parameter,

$\Omega_M = \Omega = \frac{8\pi G\rho}{3H^2}$  the density parameter,

$q = -\frac{\ddot{R}R}{\dot{R}^2}$ , the deceleration parameter,

$\Omega_{\text{vac}} = \Omega_\lambda = \lambda = \frac{\Lambda}{3H^2}$ , the (reduced) cosmological constant,

$\Omega_c = \alpha = \frac{kc^2}{H^2 R^2}$ , the curvature parameter.

Quantities are labeled by 0 when they are referred to their present value:  $\Omega_0, q_0, \dots$

E.F.L. :

$$\Omega_c = \Omega_M + \Omega_\lambda - 1$$

# Solutions (matter dominated)

$$\ddot{a} = g = -\frac{GM}{a^2} \text{ and } \rho a^3 = \text{cste}$$

from this we have derived:

$$\dot{a}^2 - \frac{8\pi G \rho a^2}{3} = \dot{a}^2 - \frac{2GM}{a} = -k c^2$$

This is exactly the equation of a test particle in the field of one mass in Newtonian theory!

$$E_c + E_p = \text{cste}$$

Solutions:

- $k = -1$  unbound hyperbolic solution
- $k = 0$  parabolic solution
- $k = +1$  bound elliptic solution

$$k = 0 \quad (P = 0)$$

$$\dot{R}^2 = \frac{8\pi G \rho R^2}{3} \quad \text{and} \quad \rho R^3 = \rho_0 R_0^3$$

First Eq. implies:

$$\Omega = \frac{8\pi G \rho}{3 H^2} = 1 = \Omega_0$$

(present-day) critical density :

$$\rho_c = \frac{3 H_0^2}{8\pi G}$$

Second Eq. implies:

$$\dot{R}^2 = \frac{8\pi G \rho_0 R_0^3}{3 R} = H_0^2 \frac{R_0^3}{R}$$

## $k = 0$ ( $P = 0$ ) **Solution:**

$$R(t) = R_0 \left( \frac{3}{2} H_0 t \right)^{2/3} = R_0 (t/t_0)^{2/3}$$

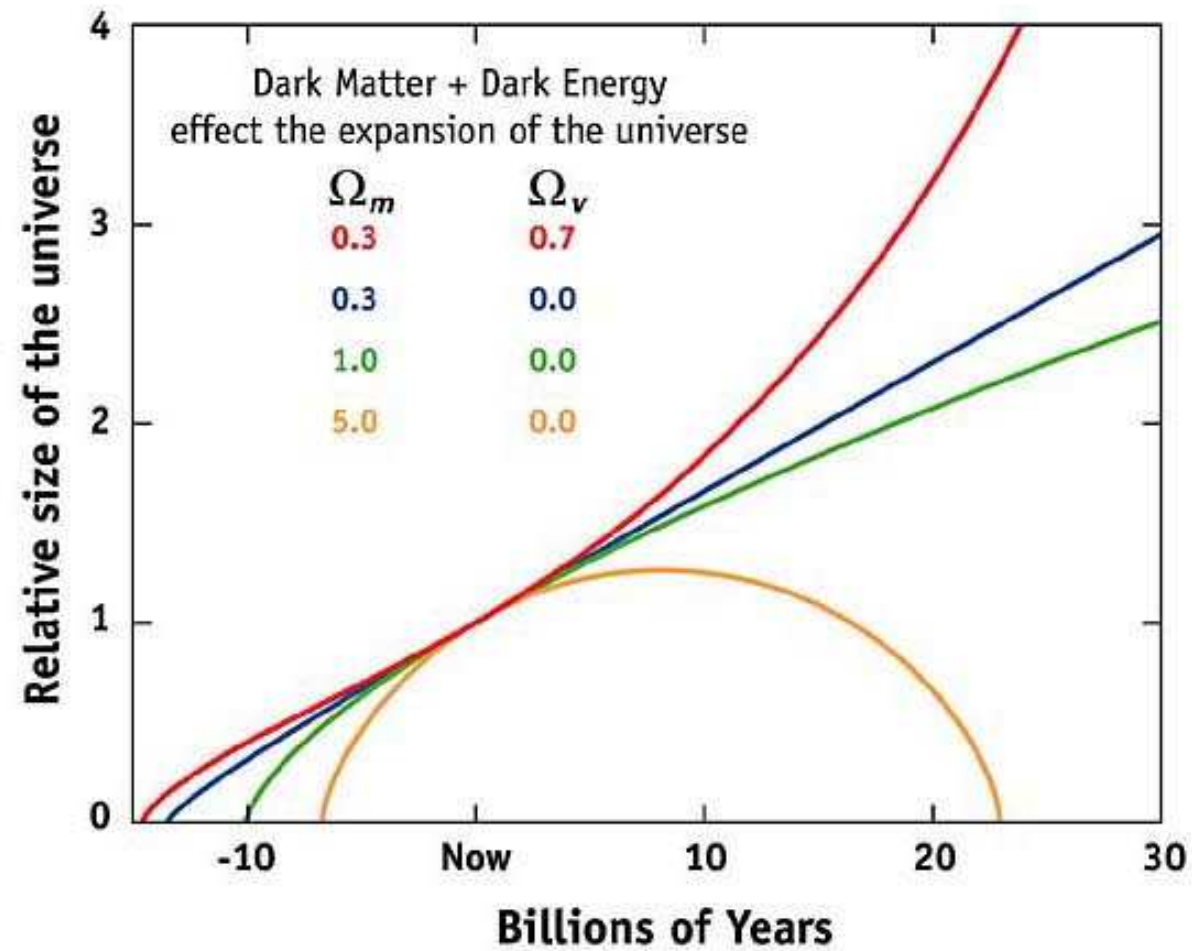
with :

$$t_0 = \frac{2}{3} H_0^{-1} = \frac{1}{\sqrt{6\pi G \rho_c}}$$

This solution goes through 0 in the past...

The solution has an “Initial” singularity.

# Solutions:



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# General behavior:

$$2\frac{\ddot{R}}{R} = -\frac{8\pi G}{3}(\rho + 3P/c^2)$$

and :

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{R^2}$$

so if:  $(\rho + 3P/c^2) > 0$   $R$  will go through 0 (in the past) in a finite time  $t_0$ .

There is a theorem more general than this.

When  $R \rightarrow 0$  than  $\left(\frac{\dot{R}}{R}\right)^2 \sim \frac{8\pi G\rho}{3}$  i.e.  $\Omega \sim 1$



# Behavior of $\Omega$ :

previous second Eq. implies  $\Omega_c = \Omega - 1$  so :

$$H^2 = H_0^2[\Omega_0(1+z)^3 + (1-\Omega_0)(1+z)^2]$$

so:

$$H^2 = H_0^2(1+z)^2(1+\Omega_0 z)$$

and:

$$\Omega(z) = \frac{8\pi G \rho}{3H^2} = \frac{8\pi G \rho_0}{3H_0^2} \frac{(1+z)^3}{(1+z)^2(1+\Omega_0 z)}$$

so:

$$\Omega(z) = \Omega_0 \frac{(1+z)}{(1+\Omega_0 z)}$$

# Mattig relation :

Along a light ray:

$$\frac{dr^2}{1 - kr^2} = \frac{c^2 dt^2}{R^2(t)} = \frac{c^2 dR^2}{R^2(t) \dot{R}^2(t)}$$

From this, setting  $v = \frac{\alpha_0}{\Omega_0 R_0} R$  in the right hand side, one can derive (...):

$$R_0 r = \frac{c}{H_0} \frac{2}{\Omega_0^2} \frac{\Omega_0(1+z) + 2 - 2\Omega_0 - (2 - \Omega_0)\sqrt{1 + \Omega_0 z}}{1+z}$$

$$\text{when } z \ll 1 \quad R_0 r \sim \frac{c}{H_0} z$$

$$\text{when } z \gg 1 \quad R_0 r \sim \frac{c}{H_0} \frac{2}{\Omega_0}$$

## $k = -1$ ( $P = 0$ ) Solution:

$$\begin{aligned}\dot{R}^2 &= \frac{8\pi G \rho R^2}{3} - kc^2 \\ &= H_0^2 \Omega_0 R_0^2 (1+z) + (1-\Omega_0) H_0^2 R_0^2\end{aligned}$$

so when  $1+z \gg \frac{1-\Omega_0}{\Omega_0}$  one has :  $R \propto t^{2/3}$

while when  $1+z \ll \frac{1-\Omega_0}{\Omega_0}$   $\dot{R} \sim cste$  one has  $R \propto t$   $R(t)$  can be developed:

$$\begin{aligned}H_0 t &= \frac{\Omega_0}{2(1-\Omega_0)^{3/2}} (\sinh(\psi) - \psi) \\ \frac{1}{1+z} &= \frac{R(t)}{R_0} = \frac{\Omega_0}{2(1-\Omega_0)} (\cosh(\psi) - 1)\end{aligned}$$

Allows analytical expression of  $H_0 t(z)$

## $k = +1$ ( $P = 0$ ) Solution:

The expression:

$$\dot{R}^2 = H_0^2 \Omega_0 R_0^2 (1 + z) + (1 - \Omega_0) H_0^2 R_0^2$$

allows to find  $R_m$  so that  $\dot{R} = 0$

$$R_m = R_0 \frac{\Omega_0}{\Omega_0 - 1}$$

$R(t)$  can be developed as well:

$$H_0 t = \frac{\Omega_0}{2(\Omega_0 - 1)^{3/2}} (\phi - \sin(\phi))$$
$$\frac{1}{1 + z} = \frac{R(t)}{R_0} = \frac{\Omega_0}{2(\Omega_0 - 1)} (1 - \cos(\phi))$$

## $k = +1$ ( $P = 0$ ) **Solution:**

At the maximum:

$$R_m = c \frac{2 t_m}{\pi}$$
$$\rho_m = \frac{3\pi}{32 G t_m^2}$$
$$t_m = \frac{1}{H_0} \frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} \pi$$

(useful for structure formation)

# Cases $\Lambda \neq 0$

$$2\ddot{R} = -\frac{8\pi G}{3}\left(\rho + \frac{3P}{c^2}\right)R + \frac{2\Lambda}{3}R$$

If  $\Lambda < 0$  it is an attractive force

If  $\Lambda > 0$  it is a repulsive force, in which case  $R(t)$  might not go through  $R = 0$ .

Case  $P = 0$

$$2\ddot{R} = H_0^2 R_0 \left[ \frac{2\lambda_0}{(1+z)} - \Omega_0 (1+z)^2 \right]$$

$$\dot{R}^2 = H_0^2 R_0^2 \left[ \frac{\lambda_0}{(1+z)^2} + (1 - \Omega_0 - \lambda_0) + \Omega_0(1+z) \right]$$

setting  $u = 1 + z$  one gets:

$$\dot{R}^2 \propto \frac{\lambda_0}{u^2} + (1 - \Omega_0 - \lambda_0) + \Omega_0 u = f(u)$$

# Case $\Lambda \neq 0, P = 0$

The “useful” relations  $R_0 r, t(z), \dots$  are not analytical.

$$\begin{aligned}\dot{R}^2 &= \frac{8 \pi G \rho R^2}{3} - kc^2 + \frac{\Lambda R^2}{3} \\ &= H_0^2 R_0^2 \left[ \frac{\Omega_\Lambda}{(1+z)^2} - \Omega_c + \Omega_0(1+z) \right]\end{aligned}$$

## Mattig relation

$$S_k^{-1}(r) = \int_{t(z)}^{t_0} \frac{c dt}{R(t)} = |\Omega_c|^{1/2} \int_1^{1+z} \frac{du}{(\Omega_0 u^3 - \Omega_c u^2 + \Omega_\Lambda)^{1/2}}$$

Age:

$$t_0 - t(z) = \int_1^{1+z} \frac{1}{H_0} \frac{du}{u(\Omega_0 u^3 - \Omega_c u^2 + \Omega_\Lambda)^{1/2}}$$

# Case $\Lambda \neq 0$ Applications

- Mattig relation :  $R_0 r(z)$
- Angular distance :  $\theta = \frac{d}{D_{ang}(z)}$   
→ minimum at some  $z$  then increases!
- Look back time:  $H_0(t_0 - t(z))$

→ at  $z \sim 1$  the universe is significantly younger:

$$\Omega \sim 0. \quad \Omega_\Lambda = 0. \quad z = 1 \leftrightarrow t_1 \sim 0.5 t_0$$

$$\Omega = 1. \quad \Omega_\Lambda = 0. \quad z = 1 \leftrightarrow t_1 \sim 0.35 t_0$$

$$\Omega = 0.3 \quad \Omega_\Lambda = 0.7 \quad z = 1 \leftrightarrow t_1 \sim 0.35 t_0$$

Models with  $(\Omega, \Omega_\Lambda > 0)$  are older than with  $(\Omega, \Omega_\Lambda = 0)$ , the difference being important only

when  $\Omega_\Lambda \sim \lambda_c$ .



# Radiation dominated

$$P = \frac{1}{3} \rho_\gamma c^2 \quad \text{and} \quad \rho_\gamma R^4 = \text{cste}$$

E.F.L. Equations:

$$\begin{aligned} \frac{\dot{R}^2}{R} &= \frac{8 \pi G}{3} (\rho_\gamma + \rho_m) - \frac{kc^2}{R^2} + \frac{\Lambda}{3} \\ &\propto \frac{1}{R^4} + \frac{1}{R^3} - \frac{1}{R^2} + \text{cste} \end{aligned}$$

→ The radiation term is dominant at high redshift:  $\dot{R} = \frac{\text{cste}}{R}$

Solution:

$$R = R_1 \left( \frac{t}{\tau} \right)^{1/2} \quad \text{with} \quad \tau = \frac{3}{32 \pi G \rho_1}$$

# Some historical remarks

**Einstein:** 1916: GR + first consistent cosmological model.

Einstein cosmological principle: The universe is homogeneous on large scale.

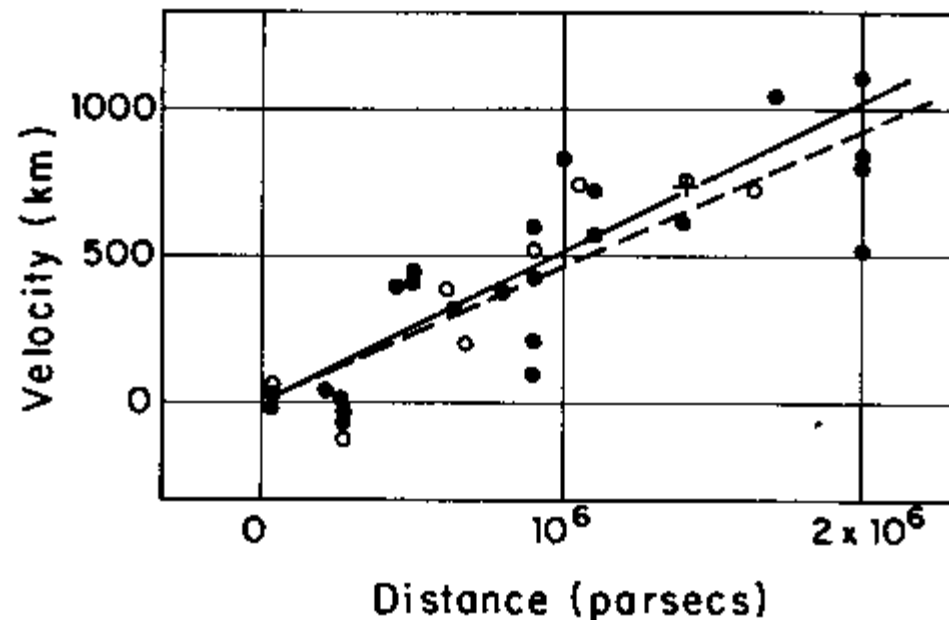
**De Sitter:** 1919 GR +  $\Lambda$  with  $\rho = 0$ .  
Static but particles move. Redshft  $\propto D$ .

**Friedman:** 1922-1924: G.R. general solutions with positive and negative curvature. Polemic with Einstein.

# Some historical remarks

**Lemaître** 1925: De Sitter world = expanding world.  
1927: expanding solution with  $\rho \neq 0$ .

**Hubble** 1929: The linear relation between  $D$  and  $v$



# 1933

**Zwicky** Missing mass in Coma.

**Lemaître** Beginning ? Singularity ? How did structures originate ?

**Gamov** 1942-1948: Origin of elements  $\rightarrow T$

**Penzias, Wilson, Dicke**'s group 1964: Discovery and interpretation of the CMB.